MATH 303 – Measure Theory
Mini-Project
Haar Measure

Instructions

- This document is divided into 4 sections. **Please submit 3 problems total, with no more than one from each section, for grading**. You are encouraged to work on additional exercises to more fully engage with the topic, but it is not required to write up any extra solutions, and there will not be any bonus points for submitting extra work.
- The problems are not all equally difficult; some are easier and others are more challenging. You are free to choose whichever problems you wish, but you are likely to learn more and have a more fulfilling experience if you attempt some of the harder exercises.
- Each problem will be graded out of 10 points and be counted with equal weight to a usual homework assignment.
- Please submit solutions as a single pdf with the exercises clearly numbered.
- Indicate the topic ("Haar measure") at the top of the first page.
- Please upload a pdf of your solutions by 23:59 on Monday, December 15.

The grade for this assignment will take into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Earlier in the course, we showed that the Lebesgue measure λ on $\mathbb{R}$ is *translation-invariant*, meaning $\lambda(E + t) = \lambda(E)$ for every Lebesgue-measurable set $E \subseteq \mathbb{R}$ and every $t \in \mathbb{R}$. For many purposes, this makes the Lebesgue measure the most "natural" measure for analysis on the real line. The goal of this mini-project is to construct translation-invariant measures (so-called *Haar measures*) on other groups and learn about some applications in representation theory and harmonic analysis.

Learning Objectives

After completing this mini-project, you will be able to:

- Reconstruct the proof of a long theorem consisting of many parts.
- Apply important theorems from the course (e.g., Riesz Representation Theorem and Fubini's Theorem) in a new context.
- Explain the role of Haar measures in representation theory and harmonic analysis.

1. Topological Groups and Haar Measures

Definition 1

A *topological group* is a group with a topology in which the group operations are continuous. Formally, G is a topological group if G is both a group and a topological space and the maps $M : G \times G \to G$ and $i : G \to G$ given by

$$M(g,h) = gh \qquad \text{and} \qquad i(g) = g^{-1}$$

are continuous.

We will be most interested in topological groups where the underlying topology is locally compact and Hausdorff. We call such groups *locally compact groups*.

Example 2

The group $(\mathbb{R}, +)$ of real numbers under addition with the standard topology is a locally compact group.

Exercise 1

(a) Write down at least 3 more examples (or families of examples) of locally compact groups.
(b) Give an example of a group that is not locally compact (when equipped with its standard topology).

Solution: There are many possible examples and non-examples of locally compact groups. Several of them are discussed on the wikipedia entry about locally compact groups: https://en.wikipedia.org/wiki/Locally_compact_group#Examples_and_counterexamples.

Definition 3

A Borel measure μ on a topological group G is said to be *left-translation-invariant* if $\mu(gE) = \mu(E)$ for every $g \in G$ and every Borel set $E \subseteq G$. A nonzero left-translation-invariant Radon measure on a locally compact group is called a *(left) Haar measure*.

Translation-invariance can also be characterized using functions instead of sets.

Proposition 4

Let G be a locally compact group, and let μ be a Radon measure on G. Then μ is a left Haar measure if and only if for every $f \in C_c(G)$ and every $y \in G$,

$$\int_G f(y^{-1}x)\, d\mu(x) = \int_G f\, d\mu.$$

Exercise 2

Prove Proposition 4.

Solution: Fix $y \in G$, and consider the Borel measure μ_y defined by $\mu_y(E) = \mu(yE)$. Since translates of open sets are open and translates of compact sets are compact, the measure μ_y is also Radon. We claim

$$\int_G f \, d\mu_y = \int_G f(y^{-1}x) \, d\mu(x) \tag{1}$$

for every $f \in L^1(\mu)$. First, if $f = \mathbb{1}_E$ for some Borel set $E \subseteq G$, then

$$\int_G \mathbb{1}_E \, d\mu_y = \mu_y(E) = \mu(yE) = \int_G \mathbb{1}_{yE}(x) \, d\mu(x) = \int_G \mathbb{1}_E(y^{-1}x) \, d\mu(x),$$

since $x \in yE \iff y^{-1}x \in E$. Hence, (1) holds for $f = \mathbb{1}_E$. By linearity of the integral, it follows that (1) holds for Borel-measurable simple functions. Then by the monotone convergence theorem and the approximability of nonnegative measurable functions by an increasing sequence of simple functions, (1) holds for all nonnegative Borel-measurable functions $f : G \to [0, \infty]$. Finally, expressing $f \in L^1(\mu)$ as $f = (f_1 - f_2) + i(f_3 - f_4)$ with $f_1, f_2, f_3, f_4 \geq 0$ and applying linearity of the integral again, we obtain (1) for every $f \in L^1(\mu)$ as claimed.

Now let us use the identity (1) to prove Proposition 4. Suppose μ is a left Haar measure. Then by definition, $\mu_y = \mu$ for every $y \in G$. Therefore,

$$\int_G f(y^{-1}x) \, d\mu(x) = \int_G f \, d\mu_y = \int_G f \, d\mu$$

for $f \in L^1(\mu)$, and in particular for $f \in C_c(G) \subseteq L^1(\mu)$.

Conversely, if

$$\int_G f(y^{-1}x) \, d\mu(x) = \int_G f \, d\mu \qquad (\forall y \in G)$$

for every $f \in C_c(G)$, then using (1), we have

$$\int_G f \, d\mu_y = \int_G f \, d\mu \qquad (\forall y \in G)$$

But μ_y and μ are both Radon measures, so by the uniqueness part of the Riesz representation theorem, $\mu_y = \mu$ for every $y \in G$. That is, μ is a left Haar measure.

Haar measures were introduced by (and subsequently named after) the Hungarian mathematician Alfréd Haar in 1933. The construction of Haar measures for general locally compact groups was given by André Weil in 1940, but the existence of Haar measures is nevertheless referred to as Haar's theorem.

Theorem 5: Haar's theorem

Let G be a locally compact group.

- EXISTENCE: There exists a left Haar measure μ on G.
- UNIQUENESS UP TO SCALING: If μ and ν are left Haar measures on G, then there exists $c \in (0, \infty)$ such that $\nu = c\mu$.

We will not give a full proof of Haar's theorem but will deal with the special case of compact groups, where a more streamlined proof is possible. Before turning to the proof, we discuss several applications of Haar's theorem in diverse areas of mathematics.

2. Applications of Haar Measures

In applications, Haar measures become particularly powerful when used in tandem with aspects of representation theory. For motivation, we begin with a focus on discrete abelian groups.

Let G be an abelian group with the discrete topology. We will write the (commutative) group operation with additive notation. The group G naturally acts on itself by translations $A_t : x \mapsto x + t$. To unlock tools from linear algebra, it is convenient to extend this action to the vector space of finitely-supported complex-valued functions on G, the so-called *group algebra* $\mathbb{C}[G] = \{f : G \to \mathbb{C} :$

$\mathrm{supp}(f)$ is finite$\}$. (Note that $\mathbb{C}[G]$ is a special case of the space of compactly supported continuous functions, so we could also write $C_c(G)$ in place of $\mathbb{C}[G]$.)

In addition to the vector space structure, the group algebra $\mathbb{C}[G]$ comes equipped with several other structures as well. First, we can extend the group operation from G to $\mathbb{C}[G]$. Consider the basis $\{\delta_x : x \in G\}$, where $\delta_x(y) = 0$ if $x \neq y$ and $\delta_x(x) = 1$. If we define an operation $\delta_x * \delta_y = \delta_{x+y}$ and extend this operation to be bilinear by writing functions $f \in \mathbb{C}[G]$ and $g \in \mathbb{C}[G]$ in the form $f = \sum_{y \in G} f(y)\delta_y$ and $g = \sum_{z \in G} g(z)\delta_z$, we have

$$f * g = \sum_{y,z \in G} f(y)g(z)\delta_{y+z} = \sum_{x \in G} \left(\sum_{y+z=x} f(y)g(z) \right) \delta_x.$$

Note that only finitely many values $f(y)$ and $g(z)$ are nonzero, so all of the sums involve only finitely many nonzero terms. This operation on $\mathbb{C}[G]$ is called the *convolution*.

Definition 6

For $f, g \in \mathbb{C}[G]$, the *convolution* is the function $f * g \in \mathbb{C}[G]$ defined by

$$(f * g)(x) = \sum_{y+z=x} f(y)g(z).$$

Second, the space $\mathbb{C}[G]$ can be equipped with various norms such as the ℓ^p-norms $\|f\|_p = \left(\sum_{x \in G} |f(x)|^p\right)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_\infty = \sup_{x \in G} |f(x)|$. Particularly notable among this family of norms is the ℓ^2-norm, since it is induced by an inner product $\langle f, g \rangle = \sum_{x \in G} f(x)\overline{g(x)}$. Moreover, the linear operators $A_t : \mathbb{C}[G] \to \mathbb{C}[G]$ defined by $(A_t f)(x) = f(x - t)$[1] are *unitary* with respect to this inner product:

$$\langle A_t f, A_t g \rangle = \sum_{x \in G} f(x-t)\overline{g(x-t)} = \sum_{y \in G} f(y)\overline{g(y)} = \langle f, g \rangle,$$

where we have used the change of variables $y = x - t$. On finite dimensional vector spaces, a classical fact from linear algebra is that unitary matrices are diagonalizable with eigenvalues of modulus 1. If G is an infinite group, then the group algebra $\mathbb{C}[G]$ is infinite dimensional. In order to recover an appropriate "diagonalization" of the operators A_t, we work with completions of $\mathbb{C}[G]$.

Definition 7

We define the *ℓ^p-spaces* by

$$\ell^p(G) = \left\{ f : G \to \mathbb{C} : \|f\|_p = \left(\sum_{x \in G} |f(x)|^p \right)^{1/p} < \infty \right\}$$

for $1 \leq p < \infty$ and

$$\ell^\infty(G) = \left\{ f : G \to \mathbb{C} : \|f\|_\infty = \sup_{x \in G} |f(x)| < \infty \right\}.$$

These ℓ^p-spaces agree with the L^p-spaces for the measure space $(G, \mathscr{P}(G), \#)$, where $\#$ is the counting measure on G. By results proved in the lecture notes, $\ell^p(G)$ is the completion of $\mathbb{C}[G]$ with respect to the ℓ^p-norm for each $p \in [1, \infty)$. The space $\ell^\infty(G)$ is also complete but is larger than the completion of $\mathbb{C}[G]$ with respect to the ℓ^∞-norm, which is the space

$$c_0(G) = \{ f : G \to \mathbb{C} : \text{for every } \varepsilon > 0, \text{ the set } \{x \in G : |f(x)| > \varepsilon\} \text{ is finite} \}.$$

[1] The minus sign in this expression is introduced so that we have a plus sign in the related expression $A_t \delta_x = \delta_{x+t}$.

By the Cauchy–Schwarz inequality,

$$\sum_{x\in G} |f(x)g(x)| \leq \|f\|_2 \|g\|_2 ,$$

so we can extend the inner product $\langle f, g\rangle = \sum_{x\in G} f(x)\overline{g(x)}$ to the space $\ell^2(G)$, and the operators A_t extend to unitary operators on $\ell^2(G)$. To obtain the "diagonalization" of the operators A_t, we consider functions in the larger space $\ell^\infty(G)$.

NOTATION. We write S^1 for the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. We give S^1 the subspace topology inherited from $\mathbb{C}$ (with the standard topology) so that S^1 becomes a compact abelian group with the operation of complex multiplication.

DEFINITION 8

A *character* on G is a group homomorphism $\chi : G \to S^1$. That is, for every $x, y \in G$,

$$\chi(x + y) = \chi(x)\chi(y).$$

EXERCISE 3

(a) Prove that every character is an eigenfunction of the translation operators A_t. That is, if $\chi : G \to S^1$ is a character and $t \in G$, show that $A_t\chi = \lambda\chi$ for some $\lambda \in \mathbb{C}$. What is the value of λ?

(b) Conversely, suppose $f \in \ell^\infty(G)$ is a nonzero function and for every $t \in G$, there exists $\lambda_t \in \mathbb{C}$ such that $A_t f = \lambda_t f$. Show that there is a character χ such that $\lambda_t = \overline{\chi(t)}$ and f is a scalar multiple of χ.

Solution: **(a)** Let $\chi : G \to S^1$ be a character. Then using the definition of A_t and then the fact that χ is a homomorphism, we have

$$(A_t\chi)(x) = \chi(x - t) = \chi(x)\chi(t)^{-1}.$$

Thus, χ is an eigenfunction of eigenvalue $\lambda = \chi(t)^{-1} = \overline{\chi(t)}$.

(b) Let $f \in \ell^\infty(G)$ such that $A_t f = \lambda_t f$ for every $t \in G$. That is,

$$f(x - t) = \lambda_t f(x) \qquad (\forall x, t \in G)$$

Then for every $x, u, v \in G$,

$$\lambda_{u+v} f(x) = f(x - u - v) = \lambda_v f(x - u) = \lambda_u\lambda_v f(x),$$

so $\lambda_{u+v} = \lambda_u\lambda_v$. Therefore, $t \mapsto \lambda_t$ is a homomorphism from G to $\mathbb{C}$. Since $f \in \ell^\infty(G)$, we also have that $\{\lambda_t : t \in G\}$ is bounded. Since $|\lambda_{nt}| = |\lambda_t|^n$ for $n \in \mathbb{Z}$, we conclude that $|\lambda_t| = 1$ for every $t \in G$, as otherwise λ would be unbounded. Homomorphisms from G to S^1 are precisely characters, so there exists a character χ such that $\lambda_t = \overline{\chi(t)}$ for every $t \in G$. Finally, we may write

$$f(x) = f(0 - (-x)) = \lambda_{-x} f(0) = f(0)\chi(x),$$

so f is a scalar multiple of χ as desired.

Since characters act as eigenfunctions for the operators A_t, it would be useful to express functions in the basis of characters. But characters do not even belong to the space $\ell^2(G)$, much less the group algebra $\mathbb{C}[G]$, so how can we represent functions using such a basis? One part of the answer comes from an algebraic result of Lev Pontryagin. The other part, as we shall see, is connected to Haar measures.

Theorem 9: Pontryagin Dual (Discrete Case)

Let G be a discrete abelian group. The family $\widehat{G}$ of characters on G is a compact abelian group, where the topology is the topology of pointwise convergence and the group operation is given by pointwise multiplication.

The group $\widehat{G}$ is called the *(Pontryagin) dual* of G. We will not prove Pontryagin's theorem in full but will illustrate it with a simple example.

Example 10

Suppose $G = \mathbb{Z}$. Since $\mathbb{Z}$ is generated by the single element 1, a character on $\mathbb{Z}$ is a function $\chi : \mathbb{Z} \to S^1$ satisfying $\chi(n) = \chi\left(\underbrace{1 + \ldots + 1}_{n \text{ times}}\right) = \chi(1)^n$. Thus, every character on $\mathbb{Z}$ is of the form $\chi(n) = \lambda^n$ for some element $\lambda \in S^1$. This allows us to identify the dual group $\widehat{\mathbb{Z}}$ with the circle group S^1. For calculations, it is often more convenient to replace this identification with S^1 by an identification with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by writing characters as $\chi(n) = e^{2\pi i \xi n}$ for some $\xi \in \mathbb{T}$.

By Haar's theorem, there is a unique translation-invariant Borel probability measure $\mu_{\widehat{G}}$ on $\widehat{G}$. This measure allows us to express functions on G as *integral combinations* of characters.

Definition 11

Let $f \in \ell^1(G)$. The *Fourier transform* of f is the function $\widehat{f} : \widehat{G} \to \mathbb{C}$ defined by $\widehat{f}(\chi) = \sum_{x \in G} f(x)\overline{\chi(x)}$.

Exercise 4

Prove that if $f \in \ell^1(G)$, then $\widehat{f}$ is a continuous function on $\widehat{G}$ and $\left\|\widehat{f}\right\|_{\sup} \leq \|f\|_1$.

Solution: Suppose $(\chi_n)_{n \in \mathbb{N}}$ is a sequence of characters such that $\chi_n \to \chi$ in $\widehat{G}$. That is, $\lim_{n\to\infty} \chi_n(x) = \chi(x)$ for every $x \in G$. Let $g_n(x) = f(x)\overline{\chi_n(x)}$ for $n \in \mathbb{N}$ and $x \in G$. Then $g_n \to f\overline{\chi}$ pointwise on G and $|g_n| = |f| \in \ell^1(G)$, so by the dominated convergence theorem,

$$\lim_{n\to\infty} \widehat{f}(\chi_n) = \lim_{n\to\infty} \sum_{x \in G} f(x)\overline{\chi_n(x)} = \sum_{x \in G} f(x)\overline{\chi}(x) = \widehat{f}(\chi).$$

This proves that $\widehat{f}$ is continuous.

Now, by the triangle inequality,

$$\left\|\widehat{f}\right\|_{\sup} = \sup_{\chi \in \widehat{G}} \left|\widehat{f}(\chi)\right| \leq \sup_{\chi \in \widehat{G}} \sum_{x \in G} |f(x)||\overline{\chi}(x)| = \sum_{x \in G} |f(x)| = \|f\|_1 .$$

Theorem 12: Fourier Inversion (Discrete Groups)

Let $f \in \ell^1(G)$. Then

$$f(x) = \int_{\widehat{G}} \widehat{f(\chi)} \chi(x)\, d\mu_{\widehat{G}}(\chi)$$

for every $x \in G$.

Exercise 5

(a) Prove the special case of Theorem 12 for the function $f = \delta_0$, i.e.

$$\int_{\widehat{G}} \chi(x)\, d\mu_{\widehat{G}}(\chi) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0. \end{cases}$$

You may use the fact (without proof) that characters separate points: if $x, y \in G$ and $x \neq y$, then there exists $\chi \in \widehat{G}$ such that $\chi(x) \neq \chi(y)$.

(b) Use part (a) to prove the general case of Theorem 12 (for arbitrary $f \in \ell^1(G)$).

Solution: **(a)** If $x = 0$, then $\chi(0) = 1$ for every $\chi \in \widehat{G}$, so

$$\int_{\widehat{G}} \chi(0) d\mu_{\widehat{G}}(\chi) = \mu_{\widehat{G}}(\widehat{G}) = 1.$$

Suppose $x \neq 0$. Let $\chi_0 \in \widehat{G}$ such that $\chi_0(x) \neq 1$. Then using invariance of the Haar measure on $\widehat{G}$, we have

$$\chi_0(x) \cdot \int_{\widehat{G}} \chi(x)\, d\mu_{\widehat{G}}(\chi) = \int_{\widehat{G}} (\chi_0\chi)(x)\, d\mu_{\widehat{G}}(\chi) = \int_{\widehat{G}} \chi(x)\, d\mu_{\widehat{G}}(\chi).$$

Hence,

$$\int_{\widehat{G}} \chi(x)\, d\mu_{\widehat{G}}(\chi)$$

is unchanged when multiplied by the number $\chi_0(x)$. Since we chose χ_0 such that $\chi_0(x) \neq 1$, this implies

$$\int_{\widehat{G}} \chi(x)\, d\mu_{\widehat{G}}(\chi) = 0.$$

(b) Using the homomorphism property for characters and part (a), we have

$$\int_{\widehat{G}} \chi(x - y) d\mu_{\widehat{G}}(\chi) = \delta_y(x)$$

for $x, y \in G$. An arbitrary function $f \in \ell^1(G)$ may be written as $f = \sum_{y \in S} f(y)\delta_y$ where $S = \{y \in G : f(y) \neq 0\}$ is a countable set by the exercise in Homework 2 (see part (e)). Enumerating S as $S = \{y_1, y_2, \ldots\}$ and applying the dominated convergence theorem, we have

$$\begin{aligned} f(x) = \lim_{N\to\infty} f_N(x) &= \lim_{N\to\infty} \sum_{n=1}^{N} f(y_n)\delta_{y_n} = \lim_{N\to\infty} \sum_{n=1}^{N} f(y_n) \int_{\widehat{G}} \chi(x - y_n)\, d\mu_{\widehat{G}}(\chi) \\ &= \lim_{N\to\infty} \int_{\widehat{G}} \sum_{n=1}^{N} f(y_n)\chi(x - y_n)\, d\mu_{\widehat{G}}(\chi) = \int_{\widehat{G}} \sum_{n=1}^{\infty} f(y_n)\chi(x - y_n)\, d\mu_{\widehat{G}}(\chi) \\ &= \int_{\widehat{G}} \sum_{y \in G} f(y)\chi(x - y)\, d\mu_{\widehat{G}}(\chi) = \int_{\widehat{G}} \sum_{y \in G} f(y)\overline{\chi(y)}\chi(x)\, d\mu_{\widehat{G}}(\chi) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(x)\, d\mu_{\widehat{G}}(\chi). \end{aligned}$$

The Fourier transform interacts nicely with other algebraic operations.

Theorem 13

Let G be a discrete abelian group.

(1) The space $\ell^1(G)$ is a Banach algebra under convolution: if $f, g \in \ell^1(G)$, then $f*g \in \ell^1(G)$ and $\|f*g\|_1 \le \|f\|_1 \|g\|_1$.

(2) Convolution identity: if $f, g \in \ell^1(G)$, then $\widehat{f*g} = \widehat{f} \cdot \widehat{g}$.

(3) Conjugation identity: if $f \in \ell^1(G)$ and $\widetilde{f}(x) = \overline{f(-x)}$, then $\widehat{\widetilde{f}}(\chi) = \widehat{\overline{f}}(\overline{\chi}) = \overline{\widehat{f}(\chi)}$.

(4) Parseval–Plancherel identity: if $f, g \in \ell^1(G)$, then

$$\sum_{x\in G} f(x)\overline{g(x)} = \int_{\widehat{G}} \widehat{f}(\chi)\overline{\widehat{g}(\chi)}\, d\mu_{\widehat{G}}(\chi).$$

Exercise 6

Prove Theorem 13. For part (4), it may be helpful to express $\widehat{f} \cdot \overline{\widehat{g}}$ as the Fourier transform of a function using parts (2) and (3).

Solution: **(1)** Given $f, g \in \ell^1(G)$, we may compute:

$$\|f*g\|_1 = \sum_{x\in G} |(f*g)(x)| = \sum_{x\in G} \left| \sum_{y\in G} f(x-y)g(y) \right|.$$

Applying the triangle inequality and then changing the order of summation (which is valid since all terms are nonnegative), we have

$$\|f*g\|_1 = \sum_{x\in G} |(f*g)(x)| \le \sum_{x\in G}\sum_{y\in G} |f(x-y)g(y)|$$
$$= \sum_{y\in G} |g(y)| \sum_{x\in G} |f(x-y)| = \sum_{y\in G} |g(y)| \sum_{z\in G} |f(z)| = \|f\|_1 \|g\|_1,$$

where we applied the change of variables $z = x - y$.

(2) Let $f, g \in \ell^1(G)$ and $\chi \in \widehat{G}$. Then

$$\widehat{f*g}(\chi) = \sum_{x\in G} (f*g)(x)\overline{\chi(x)} = \sum_{x\in G}\sum_{y\in G} f(x-y)g(y)\overline{\chi(x)}.$$

By part (a), this sum is unconditionally summable, so we may reorder the summation and apply the change of variables $z = x - y$ to obtain

$$\widehat{f*g}(\chi) = \sum_{y\in G} g(y) \sum_{x\in G} f(x-y)\overline{\chi(x)} = \sum_{y\in G} g(y) \sum_{z\in G} f(z)\overline{\chi(y+z)}$$
$$= \sum_{y\in G} g(y)\overline{\chi(y)} \sum_{z\in G} f(z)\overline{\chi(z)} = \widehat{g}(\chi)\widehat{f}(\chi).$$

(3) We compute directly:

$$\widehat{\widetilde{f}}(\chi) = \sum_{x\in G} \widetilde{f}(x)\overline{\chi(x)} = \sum_{x\in G} \overline{f(-x)\chi(x)} = \sum_{y\in G} \overline{f(y)}\chi(y).$$

If we write $\chi = \overline{\overline{\chi}}$, then we see $\widehat{\overline{\overline{f}}}(\chi) = \widehat{\overline{f}}(\overline{\chi})$. On the other hand, if we conjugate the whole sum, then

$$\widehat{\overline{\overline{f}}}(\chi) = \overline{\sum_{y \in G} f(y)\overline{\chi(y)}} = \overline{\widehat{f}(\chi)}.$$

(4) By part (3), $\overline{\widehat{g}} = \widehat{\widetilde{g}}$, so by (2), $\widehat{f * \widetilde{g}} = \widehat{f} \cdot \widehat{\widetilde{g}} = \widehat{f} \cdot \overline{\widehat{g}}$. Thus, applying Fourier inversion,

$$\int_{\widehat{G}} \widehat{f}(\chi)\overline{\widehat{g}(\chi)}\ d\mu_{\widehat{G}}(\chi) = \int_{\widehat{G}} \widehat{f * \widetilde{g}}\ d\mu_{\widehat{G}}(\chi) = (f * \widetilde{g})(0) = \sum_{x \in G} f(x)\widetilde{g}(-x) = \sum_{x \in G} f(x)\overline{g(x)}.$$

Much of the analysis from this section extends from the setting of discrete abelian groups to general locally compact abelian groups and, with appropriate modifications, also to non-abelian locally compact groups. With a non-discrete topology, the group algebra can be replaced by the space $C_c(G)$ of compactly supported continuous functions, for which we consider completions with respect to L^p norms in analogy with the discussion above. Rather than working with sums, we define the L^p norms, convolutions, etc. using integration with respect to the Haar measure. For a discrete group, the counting measure is a Haar measure, so the sums written above could in fact be rewritten as integrals for consistency with the non-discrete case. We will not reprove any results in the locally compact setting, but we summarize some important results and definitions below.

Definition 14

Let G be a locally compact group with Haar measure μ_G, and let $f, g : G \to \mathbb{C}$ be Borel-measurable functions. The *convolution* $f * g$ is given by

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\ d\mu_G(y)$$

whenever $y \mapsto f(xy^{-1})g(y)$ is integrable.

Definition 15

Let G be a locally compact abelian group with Haar measure μ_G. A *character* on G is a continuous homomorphism $\chi : G \to S^1$.

Remark. Note that to account for the topology of G, we have imposed the assumption that χ is continuous. One could impose the weaker assumption of Borel-measurability, but a result of Weil says that every measurable homomorphism is in fact continuous (a result known as *automatic continuity*), so this does not produce any new functions.

The reason characters play such a central role in the analysis of functions on G is that they are the finite-dimensional irreducible representations of G. When G is non-abelian, there are irreducible representations in higher dimensions that must be accounted for as well. We will not address these issues in this mini-project.

Theorem 16: Pontryagin Duality

Let G be a locally compact abelian group.

(1) The set of characters $\widehat{G}$ is also a locally compact abelian group with the compact-open topology[a] and the operation of pointwise multiplication.

(2) For each $x \in G$, the function $\mathrm{ev}_x : \widehat{G} \to S^1$ defined by $\mathrm{ev}_x(\chi) = \chi(x)$ is a character on $\widehat{G}$, and $x \mapsto \mathrm{ev}_x$ is a homeomorphism and group isomorphism between G and $\widehat{\widehat{G}}$.

(3) If G is discrete, then $\widehat{G}$ is compact. If G is compact, then $\widehat{G}$ is discrete.

[a] The *compact-open topology* is, informally, the topology of uniform convergence on compact sets; see https://en.wikipedia.org/wiki/Compact-open_topology.

Definition 17

Let G be a locally compact abelian group with Haar measure μ_G and $\widehat{G}$ its dual with Haar measure $\mu_{\widehat{G}}$. Given $f \in L^1(\mu_G)$, the *Fourier transform* is the function $\widehat{f} : \widehat{G} \to \mathbb{C}$ given by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, d\mu_G(x).$$

The Fourier transform in this setting retains the properties (with appropriate small adjustments) stated in Theorem 13.

One may also consider Fourier transforms of measures, and here the Haar measure takes on a special role once again. Given a measure μ on a locally compact abelian group G, characters $\chi, \chi' : G \to S^1$ may have some correlation with respect to μ given by

$$\int_G \chi \cdot \overline{\chi'}\, d\mu.$$

These correlations are captured by the *Fourier transform* of μ,

$$\widehat{\mu}(\chi) = \int_G \overline{\chi}\, d\mu.$$

As shown in Homework 5 for the special case $G = \mathbb{T}$, Fourier transforms of measures can be identified as (continuous) positive definite functions on the dual group.

Theorem 18: Bochner's Theorem

Let G be a locally compact abelian group with Pontryagin dual $\widehat{G}$. If $f : \widehat{G} \to \mathbb{C}$ is a continuous positive definition function on G, then there exists a unique finite Radon measure μ on G such that $f = \widehat{\mu}$.

When G is compact, the Haar measure is the measure corresponding to the positive definite function taking the value 1 at the identity element and 0 elsewhere. In other words, it is the unique Radon measure ensuring that characters on G are orthogonal.

3. Convolutions and Means

We now begin the proof of Haar's theorem in the special case of compact metrizable groups. In the case of a compact group G, the typical normalization for Haar measures (since they are unique up to scaling) is the probability normalization $\mu(G) = 1$. We will denote the space of Borel probability measures on G by $\mathcal{M}_1(G)$ and the space of all finite Borel measures on G by $\mathcal{M}(G)$.

> Remark. In the setting of compact metric spaces, every finite Borel measure is Radon. (We will prove this in Chapter 8 of the lecture notes.)

The notions in the following definition will play an important role in the construction of a Haar measure.

Definition 19

Let G be a compact metrizable group.

- Given two Borel probability measures $\mu, \nu \in \mathcal{M}_1(G)$, we define the *convolution* to be the measure $\mu * \nu$ defined by
$$(\mu * \nu)(E) = \int_G \nu(y^{-1}E)\ d\mu(y)$$
for Borel subsets $E \subseteq G$.
- For a continuous function $f : G \to \mathbb{C}$ and a Borel probability measure $\mu \in \mathcal{M}(G)$, we define the *convolution* by
$$(\mu * f)(x) = \int_G f(y^{-1}x)\ d\mu(y) \qquad \text{and} \qquad (f * \mu)(x) = \int_G f(xy^{-1})\ d\mu(y).$$
- We say that a number $m \in \mathbb{C}$ is a *left mean* of f if there exists a Borel probability measure $\mu \in \mathcal{M}_1(G)$ such that $\mu * f = m$. Similarly, m is a *right mean* if there is a Borel probability measure such that $f * \mu = m$.

As an illustration to develop intuition about convolutions, it is useful to consider the case that the measure μ is a discrete probability measure of the form $\mu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$ for some $n \in \mathbb{N}$ and points $y_1, \dots, y_n \in G$.

Example 20: Convolution with a finitely supported measure

Let $\mu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$ for some $n \in \mathbb{N}$ and points $y_1, \dots, y_n \in G$. Then given $f \in C(G)$, the convolution $\mu * f$ takes the form
$$(\mu * f)(x) = \int_G f(y^{-1}x) d\mu(y) = \frac{1}{n}\sum_{j=1}^n f(y_j^{-1}x).$$
That is, $\mu * f$ is an average over shifts of f by $y_1, \dots, y_n$.

From Example 20, we can see that convolution with μ "smooths out" the function f. Our general strategy for constructing a Haar measure is to average over more and more points y_j so that the smoothing eventually detects an average value of f (the left mean), which we will assign as the integral of f with respect to the Haar measure.

In order to carry out this process, we will need to utilize several properties of convolution, summarized in the next proposition.

Proposition 21

Let G be a compact group. Let μ, ν, ρ be finite Borel measures on G, and let $f, g \in C(G)$.

(1) Bilinearity: $\mu*(\nu+\rho) = \mu*\nu+\mu*\rho$, $(\mu+\nu)*\rho = \mu*\rho+\nu*\rho$, $(\mu+\nu)*f = \mu*f+\nu*f$, $f*(\mu+\nu) = f*\mu+f*\nu$, $\mu*(f+g) = \mu*f+\mu*g$, and $(f+g)*\mu = f*\mu+g*\mu$.

(2) Associativity: $\mu*(\nu*\rho) = (\mu*\nu)*\rho$, $(\mu*\nu)*f = \mu*(\nu*f)$, $f*(\mu*\nu) = (f*\mu)*\nu$, and $(\mu*f)*\nu = \mu*(f*\nu)$.

(3) Continuity: The maps $(\mu, \nu) \mapsto \mu*\nu$, $(\mu, f) \mapsto \mu*f$, and $(f, \mu) \mapsto f*\mu$ are continuous.

Exercise 7

Prove the following cases of bilinearity and associativity:
(a) $\mu * (\nu + \rho) = \mu * \nu + \mu * \rho$
(b) $f * (\mu + \nu) = f * \mu + f * \nu$
(c) $\mu * (\nu * \rho) = (\mu * \nu) * \rho$
(d) $(\mu * \nu) * f = \mu * (\nu * f)$

Solution: We verify each part by direct calculation.

(a) By linearity of the integral (with respect to μ),

$$\begin{aligned}(\mu * (\nu + \rho))(E) &= \int_G (\nu + \rho)(y^{-1}E)\; d\mu(y) \\ &= \int_G \nu(y^{-1}E)\; d\mu(y) + \int_G \rho(y^{-1}E)\; d\mu(y) = (\mu * \nu)(E) + (\mu * \rho)(E).\end{aligned}$$

(b)

$$\begin{aligned}(f * (\mu + \nu))(x) &= \int_G f(xy^{-1})\; d(\mu + \nu)(y) \\ &= \int_G f(xy^{-1})\; d\mu(y) + \int_G f(xy^{-1})\; d\nu(y) = (f * \mu)(x) + (f * \nu)(x).\end{aligned}$$

(c) On the one hand,

$$(\mu * (\nu * \rho))(E) = \int_G (\nu * \rho)(y^{-1}E)\; d\mu(y) = \int_G \left(\int_G \rho(z^{-1}y^{-1}E)\; d\nu(z)\right)\; d\mu(y).$$

On the other hand,

$$((\mu * \nu) * \rho)(E) = \int_G \rho(x^{-1}E)\; d(\mu * \nu)(x).$$

We now analyze integration with respect to the measure $\mu * \nu$. By definition,

$$\begin{aligned}\int_G \mathbb{1}_E\; d(\mu * \nu) = (\mu * \nu)(E) &= \int_G \nu(y^{-1}E)\; d\mu(y) \\ &= \int_G \left(\int_G \mathbb{1}_{y^{-1}E}(z)\; d\nu(z)\right)\; d\mu(y) = \int_G \left(\int_G \mathbb{1}_E(yz)\; d\nu(z)\right)\; d\mu(y)\end{aligned}$$

for every Borel set $E \subseteq G$. By linearity of the integral, approximation of measurable functions by simple functions, and the monotone convergence theorem, we deduce that the identity

$$\int_G f\; d(\mu * \nu) = \int_G \left(\int_G f(yz)\; d\nu(z)\right)\; d\mu(y) \tag{2}$$

holds for every nonnegative measurable function $f : G \to [0, \infty]$. In particular,

$$((\mu * \nu) * \rho)(E) = \int_G \left(\int_G \rho((yz)^{-1}E)\; d\nu(z)\right)\; d\mu(y) = \int_G \left(\int_G \rho(z^{-1}y^{-1}E)\; d\nu(z)\right)\; d\mu(y).$$

Thus, $\mu * (\nu * \rho) = (\mu * \nu) * \rho$.

(d) In part (c), we showed that (2) holds for nonnegative measurable functions. Expanding complex-valued functions as linear combinations of nonnegative functions shows that (2) consequently holds for integrable complex-valued functions, in particular for $f \in C(G)$. Therefore,

$$\begin{aligned}((\mu * \nu) * f)(x) &= \int_G f(y^{-1}x)\ d(\mu * \nu)(y) \\ &= \int_G \left(\int_G f((uv)^{-1}x)\ d\nu(v) \right) d\mu(u) = \int_G \left(\int_G f(v^{-1}u^{-1}x)\ d\nu(v) \right) d\mu(u).\end{aligned}$$

Computing directly from the definitions,

$$(\mu * (\nu * f))(x) = \int_G (\nu * f)(y^{-1}x)\ d\mu(y) = \int_G \left(\int_G f(z^{-1}y^{-1}x)\ d\nu(z) \right) d\mu(y).$$

The two resulting integrals agree (after taking $y = u, z = v$), so $(\mu * \nu) * f = \mu * (\nu * f)$ as desired.

The continuity of the map $(\mu, f) \mapsto \mu * f$ plays an important role in the construction of the Haar measure. Its proof is a tricky topological argument, which we include below.

Proof of Proposition 21(3). We will prove that $(\mu, f) \mapsto \mu * f$ is continuous, as this is the continuity property we will use in the construction of a Haar measure. Establishing continuity of the other maps follows by a similar argument.

Let $\varepsilon > 0$, let $\mu \in \mathcal{M}_1(G)$, and let $f \in C(G)$. We want to show that there is a vaguely open neighborhood U of μ and a number $\delta > 0$ such that if $\nu \in U$ and $g \in C(G)$ with $\|g - f\|_{\sup} < \delta$, then $\|\nu * g - \mu * f\|_{\sup} < \varepsilon$.

For each $x \in G$, let $f_x \in C(G)$ be the function $f_x(y) = f(y^{-1}x)$. Then $x \mapsto f_x$ is a continuous function from G to $C(G)$, so by compactness of G, there is a finite collection of points $x_1, \dots, x_n$ such that $\{f_{x_j} : 1 \le j \le n\}$ is $\frac{\varepsilon}{4}$-dense in $\{f_x : x \in G\}$.

Let U be the vaguely open set

$$U = \left\{ \nu \in \mathcal{M}_1(G) : \max_{1 \le j \le n} \left| \int_G f_{x_j}\ d\nu - \int_G f_{x_j}\ d\mu \right| < \frac{\varepsilon}{4} \right\},$$

and take $\delta = \frac{\varepsilon}{4}$.

Suppose $\nu \in U$ and $g \in C(G)$ with $\|g - f\|_{\sup} < \delta$. Let $x \in G$. Choose $j \in \{1, \dots, n\}$ such that $\left\|f_x - f_{x_j}\right\|_{\sup} < \frac{\varepsilon}{4}$. Then applying the triangle inequality,

$$\begin{aligned}|(\nu * g)(x) - (\mu * f)(x)| &= \left| \int_G g_x\ d\nu - \int_G f_x\ d\mu \right| \\ &\le \int_G \underbrace{|g_x - f_x|}_{<\delta = \frac{\varepsilon}{4}} d\nu + \int_G \underbrace{\left|f_x - f_{x_j}\right|}_{<\frac{\varepsilon}{4}} d\nu + \underbrace{\left| \int_G f_{x_j}\ d\nu - \int_G f_{x_j}\ d\mu \right|}_{<\frac{\varepsilon}{4}} + \int_G \underbrace{\left|f_{x_j} - f_x\right|}_{<\frac{\varepsilon}{4}} d\mu < \varepsilon.\end{aligned}$$

□

Lemma 22

Let G be a compact group, and let $f : G \to \mathbb{C}$ be a continuous function. Then there exists a unique constant $m(f) \in \mathbb{C}$ such that $m(f)$ is both a left and right mean of f.

Exercise 8

Prove Lemma 22 by completing the following steps:

(a) Assuming $f \in C(G)$ is real-valued, show that f has a left mean as follows:

(i) For continuous $\phi : G \to \mathbb{R}$, define the *oscillation* by $\operatorname{osc}(\phi) = \max(\phi) - \min(\phi)$. Show that the function $\mu \mapsto \operatorname{osc}(\mu * f)$ attains a minimum value on the space $\mathcal{M}_1(G)$ of Borel probability measure on G.

(ii) Suppose μ is a Borel probability measure such that $\delta = \operatorname{osc}(\mu * f) > 0$. Let $\phi = \mu * f$. Show that there is a probability measure ν such that $\operatorname{osc}(\nu * \phi) < \delta$.
(Hint: ν can be constructed as a measure on finitely many points so that convolution with ν averages functions by rotating points to places where ϕ is near its maximum value.)

(iii) Combine (i) and (ii) to show that there is a Borel probability measure μ such that $\mu * f$ is a constant function.

(b) Show that every complex-valued continuous function on G has a left mean.

(c) Let $f \in C(G)$, and let $m = \mu * f$ be a left mean of f and $m' = f * \mu$ a right mean (which exists for the same reason that a left mean does). Show that $m = m'$.

Solution: (a) (i) The function $\mu \mapsto \mu * f$ is a continuous function $\mathcal{M}_1(G)$ to $C(G)$ by Proposition 21, and $\operatorname{osc} : C(G) \to \mathbb{R}$ is also continuous, so the composition $\mu \mapsto \operatorname{osc}(\mu * f)$ is a continuous function from $\mathcal{M}_1(G)$ to $\mathbb{R}$. The space $\mathcal{M}_1(G)$ is compact (in the vague topology), so we conclude that $\mu \mapsto \operatorname{osc}(\mu * f)$ attains a minimum value by the extreme value theorem.

(ii) Write $\phi = \mu * f$, and let $a = \min(\phi)$ so that $\max(\phi) = a + \delta$. Let

$$U = \left\{ g \in G : \phi(g) > a + \frac{\delta}{2} \right\}$$

be the open subset of G where ϕ is closer to its maximum value than to its minimum value. Then by compactness, we may find a finite collection $g_1, \ldots, g_n \in G$ such that $G = U \cup g_1 U \cup g_2 U \cup \ldots \cup g_n U$. Define a probability measure $\nu = \frac{1}{n+1}(\delta_e + \sum_{i=1}^n \delta_{g_i})$. Then

$$(\nu * \phi)(x) = \frac{1}{n+1} \left(\phi(x) + \sum_{i=1}^n \phi(g_i^{-1} x) \right).$$

Clearly $\max(\nu * \phi) \leq \max(\phi) = a + \delta$. But for any $x \in X$, at least one of the values $x, g_1^{-1}x, \ldots, g_n^{-1}x$ belongs to U, so

$$(\nu * \phi)(x) \geq \frac{1}{n+1} \left(na + \left(a + \frac{\delta}{2} \right) \right) = a + \frac{\delta}{2(n+1)}.$$

Therefore,

$$\operatorname{osc}(\nu * \phi) \leq \delta - \frac{\delta}{2(n+1)} < \delta.$$

(iii) Using associativity properties of the convolution from Proposition 21, we have $\nu * \phi = (\nu * \mu) * f$ in part (ii). Hence, if $\mu \in \mathcal{M}_1(G)$ and $\operatorname{osc}(\mu * f) > 0$, then the function $\rho \mapsto \operatorname{osc}(\rho * f)$ does not achieve its minimum value at ρ. In other words, if μ is a minimizer of this function, then $\operatorname{osc}(\mu * f) = 0$. That is, $\mu * f$ is constant. Since the function attains a minimum by part (i), we can find such a measure μ.

(b) Part (a) establishes the existence of a left mean for real-valued continuous functions. Given a complex-valued continuous function $f \in C(G)$, write $f = f_1 + if_2$ with f_1, f_2 real-valued. By part (a), let $\mu_1 \in \mathcal{M}_1(G)$ such that $\mu_1 * f_1 = m_1$ is constant. Then let $\mu_2 \in \mathcal{M}(G)$ such that $\mu_2 * (\mu_1 * f_2) = m_2$ is constant. Take $\mu = \mu_2 * \mu_1 \in \mathcal{M}_1(G)$. Then

$$\mu * f_1 = \mu_2 * (\mu_1 * f_1) = \mu_2 * m_1 = m_1,$$

and

$$\mu * f_2 = m_2.$$

Thus, $\mu * f = \mu * (f_1 + if_2) = m_1 + im_2$ is constant, so $m = m_1 + im_2$ is a left mean of f.

(c) Let $\mu \in \mathcal{M}_1(G)$ such that $m = \mu * f$ is a left mean and $\nu \in \mathcal{M}_1(G)$ such that $m' = f * \nu$ is a right mean. Then

$$m = m * \nu = (\mu * f) * \nu = \mu * (f * \nu) = \mu * m' = m'$$

by the associativity properties in Proposition 21.

4. Haar's Theorem for Compact Groups

Theorem 23: Existence and uniqueness of Haar measure (compact groups)

Let G be a compact group. There exists a unique Haar probability measure μ on G, and μ is both left- and right-translation-invariant.

Exercise 9

Establish uniqueness of the Haar probability measure as follows. Let $m : C(G) \to \mathbb{C}$ be the map sending a continuous function f to its mean value (by Lemma 22). Prove that if μ is a left or right Haar probability measure on G, then $\int_G f\, d\mu = m(f)$ for every $f \in C(G)$.

Solution: Suppose μ is a left Haar probability measure on G. We will show that $\int_G f\, d\mu = m(f)$ for every $f \in C(G)$. Let $f \in C(G)$. By left-translation-invariance,

$$\int_G f(y^{-1}x)\, d\mu(x) = \int_G f\, d\mu$$

for every $y \in G$. Let $\nu \in \mathcal{M}_1(G)$ such that $\nu * f = m(f)$. Averaging over $y \in G$ with respect to the measure ν and applying Fubini's theorem, we have

$$\int_G f\, d\mu = \int_G \int_G f(y^{-1}x)\, d\mu(x)\, d\nu(y) = \int_G \underbrace{\left(\int_G f(y^{-1}x)\, d\nu(y) \right)}_{(\nu * f)(x) = m(f)} d\mu(x) = m(f).$$

A similar argument applies for right Haar probability measures.

Since the mean value $m(f)$ is unique by Lemma 22, if μ and ν are two (left or right) Haar probability measures on G, then $\int_G f\, d\mu = \int_G f\, d\nu$ for every $f \in C_c(G)$. By the uniqueness part of the Riesz representation theorem, we then conclude $\mu = \nu$.

Exercise 10

Prove that a Haar probability measure exists via the following steps. Let $m : C(G) \to \mathbb{C}$ be the map sending a continuous function f to its mean value (by Lemma 22).

(a) Show that if $f \in C(G)$ and $f \geq 0$, then $m(f) \geq 0$.

(b) Show that $m(\mathbb{1}) = 1$.

(c) Check that for every Borel probability measure μ and every continuous function $f \in C(G)$, one has $m(\mu * f) = m(f)$.

(d) Prove that $m : C(G) \to \mathbb{C}$ is a linear functional.

(e) Use the preceding parts and the Riesz representation theorem to show that there exists a (left and right) Haar probability measure on G.

Solution: **(a)** The convolution of a nonnegative function with a measure is always a nonnegative function, and $m(f) = \mu * f$ for some probability measure $\mu \in \mathcal{M}_1(G)$.

(b) Let $\mu \in \mathcal{M}_1(G)$ be an arbitrary Borel probability measure on G. Then

$$(\mu * \mathbb{1})(x) = \int_G \underbrace{\mathbb{1}(y^{-1}x)}_{=1} \, d\mu(y) = \mu(G) = 1.$$

(c) Let $\nu \in \mathcal{M}_1(G)$ such that $m(\mu * f) = \nu * (\mu * f)$. By associativity of convolution, we have $m(\mu * f) = (\nu * \mu) * f$, so $m(\mu * f)$ is a left mean of f. But the mean value is unique, so $m(\mu * f) = m(f)$.

(d) Let $f_1, f_2 \in C(G)$ and $c_1, c_2 \in \mathbb{C}$. Let $\mu_1 \in \mathcal{M}_1(G)$ such that $\mu_1 * f_1 = m(f_1)$. Then by part (c), let $\mu_2 \in \mathcal{M}_1(G)$ such that $\mu_2 * (\mu_1 * f_2) = m(\mu_1 * f_2) = m(f_2)$. For the measure $\mu = \mu_2 * \mu_1$, we then have $\mu * f_1 = m(f_1)$ and $\mu * f_2 = m(f_2)$. Hence, $\mu * (c_1 f_1 + c_2 f_2) = c_1 m(f_1) + c_2 m(f_2)$, so $c_1 m(f_1) + c_2 m(f_2)$ is the mean value of $c_1 f_1 + c_2 f_2$.

(e) Combining parts (a) and (d) and applying the Riesz representation theorem, there exists a unique Radon measure μ on G such that $m(f) = \int_G f \, d\mu$ for every $f \in C(G)$. By part (b), μ is a probability measure, since $\mu(G) = \int_G \mathbb{1} \, d\mu = m(\mathbb{1}) = 1$. It remains to check that μ is left- and right-translation-invariant. Let $y \in G$. Then for any $f \in C(G)$,

$$\int_G f(y^{-1}x) \, d\mu(x) = \int_G (\delta_y * f)(x) \, d\mu(x) = m(\delta_y * f) = m(f)$$

by part (c). This proves that μ is left-translation-invariant. Convolving with δ_y on the right similarly establishes right-translation-invariance.

Remarks

The construction of the Haar measure outlined in this mini-project is based on the lecture notes of Tao (see the section "Kronecker systems and Haar measure" in `https://terrytao.wordpress.com/2008/02/11/254a-lecture-11-compact-systems/`). Many details of the proof are omitted in those notes, and the exercises in this mini-project fill in the missing details.